Quasi-local definitions of energy in general relativity

Bjoern S. Schmekel
Theoretical Astrophysics Center
University of California, Berkeley, California 94720

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Defining energy is a surprisingly difficult problem in general relativity. For instance, the energy density of the gravitational field of a planet at a particular point could be determined by a comoving observer measuring the kinetic energy of a freely falling object. Due to the equivalence principle both the object and the observer fall at equal rates. Therefore, the observer would not assign any energy to the object. Other observers like an observer who is at rest with respect to the planet would measure different values. This raises the question of how energy depends on the choice of an observer which violates the philosophy of general relativity whose tensorial equations are independent of the used reference system.

In classical electrodynamics the stress-energy tensor is a measure of the energy and momentum transported by the electromagnetic field due to a source distribution j^μ. A similar construction in general relativity leads to the so-called Bel-Robinson tensor $T_{\mu\nu\rho\sigma}$ [1, 2, 3] which can be thought of as being induced by a stress-energy tensor $T_{\mu\nu}$. Its physical meaning however remains unknown since it does not even have units of energy density. This is a consequence of the equivalence principle which equates the gravitational mass (the "charge" of gravity) with the inertial mass. The source term, i.e. j_μ in electrodynamics and $T_{\mu\nu}$ in general relativity, does not contain the energy of the gravitational field. However, since the equations of general relativity are non-linear there may be a non-linear contribution to the stress-energy. For instance, gravitational waves do not pass through each other without distortion.

Due to the absence of Stokes theorem for second ranked tensors conserved quantities do not exist.

Landau and Lifshitz were able to prove that the stress-energy-momentum pseudotensor

$$16\pi G t^{\mu\nu} = (-g)^{-1}\left[(-g)\left(g^{\mu\nu}g^{\rho\sigma} - g^{\mu\rho}g^{\nu\sigma}\right)\right]_{,\rho\sigma} - 2G^{\mu\nu} \tag{1}$$

is the only symmetric pseudotensor constructed only from the metric such that the four-divergence of the total stress energy vanishes like $\left[(-g)(T^{\mu\nu}+t^{\mu\nu})\right]_{,\mu} =$

0 and which also vanishes locally in an inertial frame. The latter requirement is dictated by the equivalence principle as was mentioned above. However, $t^{\mu\nu}$ still does not transform as a tensor.

Because of the problems associated with defining a local energy density it may be easier to make sense of the energy enclosed by a boundary. For regions of finite extend we expect non-zero values because in general a coordinate transformation can make the connection coefficients vanish at only one point.

Therefore, it seems the only sensible way to define energy is by defining energy itself and not energy density. Of course this may seem ugly because a local covariant and tensorial formulation depends on densities evaluated at a point and its infinitesimally small neighborhood (in order to compute derivatives). A point remains a point under a Lorentz transformation, but needless to say the size of a finite region depends on the observer, so obviously such an energy will depend on the chosen coordinate system. It is therefore maybe not surprising that the first useful notions of energy were defined at infinity, i.e. they enclosed the whole system (cf. ADM mass [4], Bondi mass [5]). Like a point an infinitely large box does not change its size under a change of observer.

A successful definition of quasi-local energy (QLE) was given by Brown and York [6]. A spacelike three-dimensional hypersurface Σ is embedded in a four-dimensional spacetime M which satisfies the Einstein field equations. This embedding defines the "time-direction". Finally, a two-dimensional boundary B, which encloses the energy of the region we are interested in, is embedded in the three-dimensional hypersurface Σ. Σ is enclosed by a three-boundary ${}^3B$, and their normals are constrained to be perpendicular to each other. This restriction ensures the time evolution of the system is consistent with the presence of the fixed boundary B. In classical mechanics the Hamiltonian is given by the variation of the classical action with respect to the endpoints times minus one. The Brown-York QLE is derived by considering the change of the classical action under a displacement of the initial and final hypersurfaces and is given by

$$E = \frac{1}{\kappa} \int_B d^2x \sqrt{\sigma}\,(k - k_0) \tag{2}$$

where k is the trace of the extrinsic curvature of B and σ is the metric of B. The surface gravity is denoted by κ, and k_0 is a reference term which sets the energy of flat space to zero. The subtraction term has been criticized. However, it should be mentioned that the ADM energy makes reference to flat space as well by using ordinary non-covariant derivatives.

For a Schwarzschild black hole the action can be expressed in terms of the QLE as follows

$$S = \frac{8\pi}{\kappa} \int (N dt)(rf) \tag{3}$$

where $-rf$ is the unreferenced QLE. The metric has been expressed as $ds^2 = -N^2(r)dt^2 + f^{-2}(r) + r^2 d\Omega^2$ where $N(r) = f(r) = \sqrt{1 - 2m/r}$. Because the

geodesics of infalling objects can be determined both inside and outside of a black hole it should be possible to assign a value to the energy of the gravitational field in both regions. The definition of E can be continued into the region inside the horizon [7]. Since both N and f become imaginary inside the horizon $-rf$ needs to be multiplied by i in order to become real.

Whether the quantity defined above is useful depends on its properties and whether applications exist. At infinity the QLE is equal to the ADM energy. Furthermore, it reduces to the Newtonian binding energy in the non-relativistic limit. In the thermodynamics of black holes the QLE is just the total energy. Blau and Rollier have shown that the extended Brown-York energy describes the effective potential of a particle falling into a black hole [8]. Also worth mentioning is the small sphere limit [9]. In this limit the QLE reduces to the energy of the enclosed matter. The gravitational binding energy only shows up in higher orders of the radius which emphasizes the fact that one cannot make sense of the energy of the gravitational field locally. This may be seen as a hint that point particles do not exist [7].

The most serious drawback is that fact that not all physically interesting boundaries B can be embedded in a reference space which is typically taken to be $\mathcal{R}^3$ leading to a non-existence of the reference term. An important example is the horizon of a Kerr black hole. Usually only energy differences are important, so the absence of the reference term might not lead to problems. However, the stability of flat space rests on the fact that every non-flat spacetime contains more energy that the flat ground state [10]. Therefore, Epp [11] and subsequently Liu und Yau [12] considered a modification of the Brown-York energy which does not need the three-boundary ${}^3B$. Rather, the two-boundary B is embedded directly into the four-dimensional spacetime M, and the orthogonality condition is not applicable anymore. Such an embedding does always exist. However, it is not unique. The absence of the orthogonality condition implies that the observer is not at rest anymore with respect to B. Denoting the trace of the normal momentum surface density by l which measures the expansion of B in time the boost-invariant QLE becomes

$$E = \frac{1}{\kappa} \int_B d^2x \sqrt{\sigma} \left(\sqrt{k^2 - l^2} - \sqrt{k_0^2 - l_0^2} \right) \tag{4}$$

The most attractive feature of the boost-invariant QLE is certainly its independence of the observer. It can attain complex values if B is located inside the event horizon of a black hole. Also, the integrand of the unreferenced boost-invariant QLE is always positive (if real), whereas the Brown-York QLE depends on the extrinsic curvature of B which can be positive or negative.

Ultimately, the usefulness of the described quantities will depend on the availability of applications. A more thorough review of the problems associated with quasi-local energy can be found in [13].

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References

[1] L Bel. *CR Acad. Sci. Paris*, **247**, 1094 (1958).

[2] I. Robinson. *Kings College Lectures, unpublished*. 1958.

[3] L Bel. *CR Acad. Sci. Paris*, **248**, 1297 (1959).

[4] R. Arnowitt, S. Deser, and C. W. Misner. The dynamics of general relativity. In L. Witten, editor, *Gravitation: An Introduction to Current Research*, page 246, New York, 1962. Wiley.

[5] A. Trautman, F. A. E. Pirania, and H. Bondi. *Lectures on General Relativity*, volume 1. Prentice-Hall, Englewood Cliffs, NJ, 1965. Brandeis 1964 Summer Institute on Theoretical Physics.

[6] J. David Brown and James W. York, Jr. Quasilocal energy and conserved charges derived from the gravitational action. *Phys. Rev.*, **D47**, 1407–1419 (1993).

[7] Andrew P. Lundgren, Bjoern S. Schmekel, and James W. York, Jr. Self-renormalization of the classical quasilocal energy. *Phys. Rev. D*, **75**, 084026 (2007), gr-qc/0610088.

[8] Matthias Blau and Blaise Rollier. Brown-york energy and radial geodesics. arXiv:0708.0321 [gr-qc].

[9] J. David Brown, S. R. Lau, and Jr. York, James W. Canonical quasilocal energy and small spheres. *Phys. Rev.*, **D59**, 064028 (1999), gr-qc/9810003.

[10] Richard Schon and Shing-Tung Yau. Proof of the positive mass theorem. 2. *Commun. Math. Phys.*, **79**, 231–260 (1981).

[11] Richard J. Epp. Angular momentum and an invariant quasilocal energy in general relativity. *Phys. Rev. D*, **62**, 124018 (2000), gr-qc/0003035.

[12] Chiu-Chu Melissa Liu and Shing-Tung Yau. Positivity of quasilocal mass. *Phys. Rev. Lett.*, **90**, 231102 (2003), gr-qc/0303019.

[13] Laszlo B. Szabados. Quasi-local energy-momentum and angular momentum in GR: A review article. *Living Reviews in Relativity*, **7**.